

# ON THE SPECTRAL DISTRIBUTIONS OF DISTANCE- $k$ GRAPH OF FREE PRODUCT GRAPHS.

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**ABSTRACT.** We calculate the distribution with respect to the vacuum state of the distance- $k$  graph of a  $d$ -regular tree. From this result we show that the distance- $k$  graph of a  $d$ -regular graphs converges to the distribution of the distance- $k$  graph of a regular tree. Finally, we prove that, properly normalized, the asymptotic distributions of distance- $k$  graphs of the  $d$ -fold free product graph, as  $d$  tends to infinity, is given by the distribution of  $P_k(s)$ , where  $s$  is a semicircle random variable and  $P_k$  is the  $k$ -th Chebychev polynomial.

## 1. INTRODUCTION

In this paper we consider three problems on the distance- $k$  graphs, which generalize results of Kesten [11] (on random walks on free groups), McKay [13] (on the asymptotic distribution of  $d$ -regular graphs) and the free central limit of Voiculescu [15]. The first one is finding, for fixed  $d$ , the distribution w.r.t. the vacuum state of the distance- $k$  graphs of a  $d$ -regular tree. Then we consider two related problems which are in the asymptotic regime. On one hand, we show that the asymptotic distributions of distance- $k$  graphs of  $d$ -fold free product graphs, as  $d$  tends to infinity, are given by the distribution of  $P_k(s)$ , where  $s$  is a semicircle distribution and  $P_k$  is the  $k$ -th Chebychev polynomial. On the other hand, we find the asymptotic spectral distribution of the distance- $k$  graph of a random  $d$ -regular graph of size  $n$ , as  $n$  tends to infinity.

More precisely our first result is the following.

**Theorem 1.1.** *For  $d \geq 2$ ,  $k \geq 1$ , let  $A_d^{[k]}$  be the adjacency matrix of distance- $k$  graph of the  $d$ -regular tree. Then the distribution with respect to the vacuum state of  $A_d^{[k]}$  is given by the probability distribution of*

$$T_k(b) = \sqrt{\frac{d-1}{d}} P_k\left(\frac{b}{2\sqrt{d-1}}\right) - \frac{1}{\sqrt{d(d-1)}} P_{k-2}\left(\frac{b}{2\sqrt{d-1}}\right),$$

where  $P_k$  is the Chebyshev polynomial of order  $k$  and  $b$  is a random variable with Kesten-McKay distribution,  $\mu_d$ .

The spectrum of the distance- $k$  graph of the Cartesian product of graphs was first studied by Kurihara and Hibino [10] where they consider the distance-2 graph of  $K_2 \times \cdots \times K_2$  (the  $n$ -dimensional hypercube). More recently, in a series of papers [7, 8, 9, 10, 12, 14] the asymptotic spectral distribution of the distance- $k$  graph of the  $N$ -fold power of the Cartesian product was studied. These investigations, finally lead to the following theorem which generalizes the central limit theorem for Cartesian products of graphs.

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**Theorem 1.2** (Hibino, Lee and Obata [8]). *Let  $G = (V, E)$  be a finite connected graph with  $|V| \geq 2$ . For  $N \geq 1$  and  $k \geq 1$  let  $G^{[N,k]}$  be the distance- $k$  graph of  $G^N = G \times \cdots \times G$  ( $N$ -fold Cartesian power) and  $A^{[N,k]}$  its adjacency matrix. Then, for a fixed  $k \geq 1$ , the eigenvalue distribution of  $N^{-k/2} A^{[N,k]}$  converges in moments as  $N \rightarrow \infty$  to the probability distribution of*

$$(1.1) \quad \left( \frac{2|E|}{|V|} \right)^{k/2} \frac{1}{k!} \tilde{H}_k(g),$$

where  $\tilde{H}_k$  is the monic Hermite polynomial of degree  $k$  and  $g$  is a random variable obeying the standard normal distribution  $\mathcal{N}(0, 1)$ .

In the same spirit, in [2], we consider the analog of Theorem 1.2 by changing the Cartesian product by the star product.

**Theorem 1.3** (Arizmendi and Gaxiola [2]). *Let  $G = (V, E, e)$  be a locally finite connected graph and let  $k \in \mathbb{N}$  be such that  $G^{[k]}$  is not trivial. For  $N \geq 1$  and  $k \geq 1$  let  $G^{[*N,k]}$  be the distance- $k$  graph of  $G^{*N} = G \star \cdots \star G$  ( $N$ -fold star power) and  $A^{[*N,k]}$  its adjacency matrix. Furthermore, let  $\sigma = V_e^{[k]}$  be the number of neighbors of  $e$  in the distance- $k$  graph of  $G$ , then the distribution with respect to the vacuum state of  $(N\sigma)^{-1/2} A^{[*N,k]}$  converges in distribution as  $N \rightarrow \infty$  to a centered Bernoulli distribution. That is,*

$$\frac{A^{[*N,k]}}{\sqrt{N\sigma}} \longrightarrow \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1,$$

weakly.

Our second theorem is the free counterpart of the theorems above.

**Theorem 1.4.** *Let  $G = (V, E, e)$  be a finite connected graph and let  $k \in \mathbb{N}$ . For  $N \geq 1$  and  $k \geq 1$  let  $G^{[*N,k]}$  be the distance- $k$  graph of  $G^{*N} = G * \cdots * G$  ( $N$ -fold free power) and  $A^{[*N,k]}$  its adjacency matrix. Furthermore, let  $\sigma$  be the number of neighbors of  $e$  in the graph  $G$ . Then the distribution with respect to the vacuum state of  $(N\sigma)^{-k/2} A^{[*N,k]}$  converges in moments (and then weakly) as  $N \rightarrow \infty$  to the probability distribution of*

$$(1.2) \quad P_k(s),$$

where  $P_k$  is the Chebychev polynomial of order  $k$  and  $s$  is a random variable obeying the semicircle law.

Finally, our third theorem considers the asymptotic spectral distribution of the distance- $k$  graph of  $d$ -regular random graphs.

**Theorem 1.5.** *Let  $d, k$  be fixed integers and, for each  $n$ , let  $F_n(x)$  be the expected eigenvalue distribution of the distance- $k$  graph of a random regular graph with degree  $d$  and order  $2n$ . Then, as  $n$  tends to infinity,  $F_n(x)$  converges to the distribution of  $A_d^{[k]}$  with respect to the vacuum state, described in Theorem 1.1.*

Apart from this introduction the paper is organized as follows. In Section 2 we give the basic preliminaries on graphs, orthogonal polynomials and Non-Commutative Probability and Kesten-McKay distributions. Section 3 is devoted to prove Theorem 1.3. We prove Theorem 1.4 in Sections 4 and 5. Section 4 considers the case  $k = 2$ , while Section 5 considers the case  $k \geq 3$ . Finally, in Section 6 we use the results of Section 3 to prove Theorem 1.5.

## 2. PRELIMINARIES

In this section we give very basic preliminaries on graphs, free product graphs, orthogonal polynomials, Jacobi parameters and non-commutative probability. The reader familiar with these objects may skip this section.

**2.1. Graphs.** By a *rooted graph* we understand a pair  $(\mathcal{G}, e)$ , where  $\mathcal{G} = (V, E)$ , is a undirected graph with set of vertices  $V = V(\mathcal{G})$ , and the set of edges  $E = E(\mathcal{G}) \subseteq \{(x, x') : x, x' \in V, x \neq x'\}$  and  $e \in V$  is a distinguished vertex called the *root*. For rooted graphs we will use the notation  $V^0 = V \setminus \{e\}$ . Two vertices  $x, x' \in V$  are called *adjacent* if  $(x, x') \in E$ , i.e. vertices  $x, x'$  are connected with an edge. Then we write  $x \sim x'$ . Simple graphs have no loops, i.e.  $(x, x) \notin E$  for all  $x \in V$ . A graph is called *finite* if  $|V| < \infty$ . The *degree* of  $x \in V$  is defined by  $\kappa(x) = |\{x' \in V : x' \sim x\}|$ , where  $|I|$  stands for the cardinality of  $I$ . A graph is called *locally finite* if  $\kappa(x) < \infty$  for every  $x \in V$ . It is called *uniformly locally finite* if  $\sup\{\kappa(x) : x \in V\} < \infty$ .

We define the *free product* of the rooted vertex sets  $(V_i, e_i)$ ,  $i \in I$ , where  $I$  is a countable set, by the rooted set  $(\ast_{i \in I} V_i, e)$ , where

$$\ast_{i \in I} V_i = \{e\} \cup \{v_1 v_2 \cdots v_m : v_k \in V_{i_k}^0, \text{ and } i_1 \neq i_2 \neq \cdots \neq i_m, m \in \mathbb{N}\},$$

and  $e$  is the empty word.

**Definition 2.1.** The *free product of rooted graph*  $(\mathcal{G}_i, e_i)$ ,  $i \in I$ , is defined by the rooted graph  $(\ast_{i \in I} \mathcal{G}_i, e)$  with vertex set  $\ast_{i \in I} V_i$  and edge set  $\ast_{i \in I} E_i$ , defined by

$$\ast_{i \in I} E_i := \{(vu, v'u) : (v, v') \in \bigcup_{i \in I} E_i \text{ and } u, vu, v'u \in \ast_{i \in I} V_i\}.$$

We denote this product by  $\ast_{i \in I} (\mathcal{G}_i, e_i)$  or  $\ast_{i \in I} \mathcal{G}$  if no confusion arises. If  $I = [n]$ , we denote by  $G^{\ast n} = (\ast_{i \in I} G, e)$ .

Notice that for a fixed word  $u = v_1 v_2 \cdots v_m$  with  $j \in I$  with  $v_1 \notin V_j$  the subgraph of  $(\ast_{i \in I} \mathcal{G}_i, e)$  induced by the vertex set  $\{wu : w \in V_j\}$  is isomorphic to  $G_j$ . This motivates the following definition

**Definition 2.2.** If  $x, y \in \ast_{i \in I} V_i$ , we say that  $x$  and  $y$  are in the same copy of  $G_i$  if  $x = vu$  and  $y = v'u$  for some  $u \in \ast_{i \in I} V_i$  and  $v, v' \in V_j^0$  for some  $j \in I$ .

For a given graph  $G = (V, E)$ , its *distance- $k$  graph*  $G^{[k]} = (V, E^{[k]})$  is defined by

$$E^{[k]} = \{(x, y) : x, y \in V, \partial_G(x, y) = k\}.$$

For  $x \in V$ , let  $\delta(x)$  be the indicator function of the one-element set  $\{x\}$ . Then  $\{\delta(x), x \in V\}$  is an orthonormal basis of the Hilbert space  $l^2(V)$  of square integrable functions on the set  $V$ , with the usual inner product.

The *adjacency matrix*  $A = A(\mathcal{G})$  of  $\mathcal{G}$  is a 0-1 matrix defined by

$$(2.1) \quad A_{x, x'} = \begin{cases} 1 & \text{if } x \sim x' \\ 0 & \text{otherwise.} \end{cases}$$

We identify  $A$  with the densely defined symmetric operator on  $l^2(V)$  defined by

$$(2.2) \quad A\delta(x) = \sum_{x \sim x'} \delta(x')$$

for  $x \in V$ . Notice that the sum on the right-hand-side is finite since our graph is assumed to be locally finite. It is known that  $A(\mathcal{G})$  is bounded if and only if  $\mathcal{G}$  is

uniformly locally finite. If  $A(\mathcal{G})$  is essentially self-adjoint, its closure is called the *adjacency operator* of  $\mathcal{G}$  and its spectrum is called the spectrum of  $\mathcal{G}$ .

The unital algebra generated by  $A$ , i.e. the algebra of polynomials in  $A$ , is called the *adjacency algebra* of  $\mathcal{G}$  and is denoted by  $\mathcal{A}(\mathcal{G})$  or simply  $\mathcal{A}$ .

**2.2. Orthogonal Polynomials and The Jacobi Parameters.** Let  $\mu$  be a probability measure with all moments, that is  $m_n(\mu) := \int_{\mathbb{R}} |x|^n \mu(dx) < \infty$ . The Jacobi parameters  $\gamma_m = \gamma_m(\mu) \geq 0, \beta_m = \beta_m(\mu) \in \mathbb{R}$ , are defined by the recursion

$$xQ_m(x) = Q_{m+1}(x) + \beta_m Q_m(x) + \gamma_{m-1} Q_{m-1}(x),$$

where the polynomials  $Q_{-1}(x) = 0, Q_0(x) = 1$  and  $(Q_m)_{m \geq 0}$  is a sequence of orthogonal monic polynomials with respect to  $\mu$ , that is,

$$\int_{\mathbb{R}} Q_m(x) Q_n(x) \mu(dx) = 0 \quad \text{if } m \neq n.$$

**Example 2.3.** The Chebyshev polynomials of the second kind are defined by the recurrence relation

$$P_0(x) = 1, \quad P_1(x) = x,$$

and

$$(2.3) \quad xP_n(x) = P_{n+1}(x) + P_{n-1}(x) \quad \forall n \geq 1.$$

These polynomials are orthogonal with respect to the semicircular law, which is defined by the density

$$d\mu = \frac{1}{2\pi} \sqrt{4 - x^2} dx.$$

The Jacobi parameters of  $\mu$  are  $\beta_m = 0$  and  $\gamma_m = 1$  for all  $m \geq 0$ .

**2.3. Non-Commutative Probability Spaces.** A  $C^*$ -probability space is a pair  $(\mathcal{A}, \varphi)$ , where  $\mathcal{A}$  is a unital  $C^*$ -algebra and  $\varphi : \mathcal{A} \rightarrow \mathbb{C}$  is a positive unital linear functional. The elements of  $\mathcal{A}$  are called (non-commutative) random variables. An element  $a \in \mathcal{A}$  such that  $a = a^*$  is called self-adjoint.

The functional  $\varphi$  should be understood as the expectation in classical probability.

For  $a_1, \dots, a_k \in \mathcal{A}$ , we will refer to the values of  $\varphi(a_{i_1} \cdots a_{i_n})$ ,  $1 \leq i_1, \dots, i_n \leq k$ ,  $n \geq 1$ , as the *joint moments* of  $a_1, \dots, a_k$ . If there exists  $1 \leq m, l \leq n$  with  $i(m) \neq i(l)$  we call it a *mixed moment*.

For any self-adjoint element  $a \in \mathcal{A}$  there exists a unique probability measure  $\mu_a$  (its spectral distribution) with the same moments as  $a$ , that is,

$$\int_{\mathbb{R}} x^k \mu_a(dx) = \varphi(a^k), \quad \forall k \in \mathbb{N}.$$

We say that a sequence  $a_n \in \mathcal{A}_n$  converges in distribution to  $a \in \mathcal{A}$  if  $\mu_{a_n}$  converges in distribution to  $\mu_a$ . In this setting convergence in distribution is replaced by convergence in moments. Let  $(\phi_n, \mathcal{A}_n)$  be a sequence of  $C^*$ -probability spaces and let  $a \in (\mathcal{A}, \varphi)$  be a selfadjoint random variable. We say that the sequence  $a_n \in (\phi_n, \mathcal{A}_n)$  of selfadjoint random variables converges to  $a$  in moments if

$$\lim_{n \rightarrow \infty} \phi_n(a_n^k) = \phi(a^k) \text{ for all } k \in \mathbb{N}.$$

If  $a$  is bounded then convergence in moments implies convergence in distribution.

The following proposition is straightforward and will be used frequently in the paper. A sequence of polynomials  $\{P_n = \sum_{i=0}^l c_{n,i} x^i\}_{n \geq 0}$  of degree at most  $l \geq k$

is said to converge to a polynomial  $P = \sum_{i=0}^k c_i x^i$  of degree  $k$  if  $c_{i,n} \rightarrow c_i$  for  $0 \leq i \leq k$  and  $c_{i,n} \rightarrow 0$  for  $k < i \leq l$ .

**Proposition 2.4.** *Suppose that the sequence of random variables  $\{a_n\}_{n>0}$  converges in moments to  $a$  and the sequence of polynomials  $\{P_n\}_{n>0}$  converges to  $P$ . Then the random variables  $P_n(a_n)$  converges to  $P(a)$ .*

In this work we will only consider the  $C^*$ -probability spaces  $(\mathcal{M}_n, \varphi_1)$ , where  $\mathcal{M}_n$  is the set of matrices of size  $n \times n$  and for a matrix  $M \in \mathcal{M}_n$  the functional  $\varphi_1$  evaluated in  $M$  is given by

$$\varphi_1(M) = M_{11}.$$

Let  $G = (V, E, 1)$  be a finite rooted graph with vertex set  $\{1, \dots, n\}$  and let  $A_G$  be the adjacency matrix. We denote by  $A(G) \subset \mathcal{M}_n$  be the adjacency algebra, i.e., the  $*$ -algebra generated by  $A_G$ .

It is easy to see that the  $k$ -th moment of  $A$  with respect to the  $\varphi_1$  is given the the number of walks in  $G$  of size  $k$  starting and ending at the vertex 1. That is,

$$\varphi_1(A^k) = |\{(v_1, \dots, v_k) : v_1 = v_k = 1 \text{ and } (v_i, v_{i+1}) \in E\}|.$$

Thus one can get combinatorial information of  $G$  from the values of  $\varphi_1$  in elements of  $A(G)$  and vice versa.

Let us recall the free central limit theorem for free product of graphs (see, e.g. [1]) which follows from the usual free central limit theorem for random variables [15].

**Theorem 2.5** (Free Central Limit Theorem for Graphs). *Let  $G = (V, E, e)$  be a finite connected graph. Let  $A_N$  be the adjacency matrix of the  $N$ -fold free power  $G^{*N}$ , and let  $\sigma$  be the number of neighbors of  $e$  in the graph  $G$ . Then the distribution with respect to the vacuum state of  $(N\sigma)^{-1/2} A_N$  converges in moments (and thus weakly) as  $N \rightarrow \infty$  to the semicircular law.*

For the rest of the paper we define an order which will become handy when estimating vanishing terms in Sections 4 and 5.

**Definition 2.6.** Let  $A$  and  $B$  be matrices (possibly infinite), we define the order  $A \succeq B$  if  $A_{ij} \geq B_{ij}$  for all entries  $ij$ .

**Remark 2.7.** 1)  $\varphi_1(A^k) \geq \varphi_1(B^k)$  if  $A \succeq B$ .

2) For  $G_1$  and  $G_2$  graphs with  $n$  vertices,  $G_2$  is a subgraph of  $G_1$  iff  $A_{G_1} \succeq A_{G_2}$ .

3) If  $A \succeq B$  and  $C \succeq D$  implies  $AC \succeq BD$ .

**2.4. Kesten-McKay Distribution.** As we know, by the free central limit theorem, if we have a sequence of  $d$ -regular trees, then the limiting spectral distribution of the sequence, as  $d \rightarrow \infty$ , converges to a semicircular law. However, if  $d$  is fixed, and we consider a sequence of  $d$ -regular graphs, such that the number of vertices tends to infinity, then the limiting spectral distribution is not semicircular. These limiting spectral distributions, which are known as the Kesten-McKay distributions, were found by McKay [13] while studying properties of  $d$ -regular graphs and by Kesten [11] in his works on random walk on (free) groups.

Let  $d \geq 2$  be an integer, we define Kesten-McKay distribution,  $\mu_d$ , by the density

$$(2.4) \quad d\mu_d = \frac{d\sqrt{4(d-1) - x^2}}{2\pi(d^2 - x^2)} dx.$$

The orthogonal polynomials and the Jacobi parameters of these distributions are well known. More precisely, for  $d \geq 2$ , the polynomials defined by

$$T_0(x) = 1, \quad T_1(x) = x,$$

and the recurrence formula

$$(2.5) \quad xT_k(x) = T_{k+1}(x) + (d-1)T_{k-1}(x),$$

are orthogonal with respect to the distribution  $\mu_d$ . Thus, it follows that the Jacobi parameters of  $\mu_d$  are given by

$$\beta_m = 0, \quad \forall m \geq 0 \quad \text{and} \quad \gamma_0 = d, \quad \gamma_n = d-1 \quad \forall n \geq 1.$$

**Remark 2.8.** If we define the following polynomials

$$\tilde{T}_k(x) = \begin{cases} 1, & k = 0 \\ \sqrt{\frac{d-1}{d}} P_k(x) - \frac{1}{\sqrt{d(d-1)}} P_{k-2}(x), & k = 1, 2, 3, \dots, \end{cases}$$

then,  $T_k(x) = \tilde{T}_k(x/2\sqrt{d-1})$ .

In Section 6 we will generalize the following theorem due to McKay [13] which gives a connection between large  $d$ -regular graphs and Kesten-McKay distributions.

**Theorem 2.9.** *Let  $X_1, X_2, \dots$  be a sequence of regular graphs with degree  $d \geq 2$  such that  $n(X_i) \rightarrow \infty$  and  $c_k(X_i)/n(X_i) \rightarrow 0$  as  $i \rightarrow \infty$  for each  $k \geq 3$ , where  $n(X_i)$  is the order of  $X_i$  and  $c_k(X_i)$  is the number of  $k$ -cycles in  $X_i$ . Then, the limiting distribution for the eigenvalues  $X_i$  as  $i \rightarrow \infty$  is given by  $\mu_d$ .*

### 3. DISTANCE- $k$ GRAPH OF $d$ -REGULAR TREES

The  $d$ -regular tree is the  $d$ -fold free product graph of  $K_2$ , the complete graph with two vertices. Before we consider asymptotic behavior of the general case of the free product of graphs, we study the distance- $k$  graph of a  $d$ -regular tree for fixed  $d$  and  $k$ . This is an example where we can find the distribution with respect to the vacuum state in a closed form. Moreover, this example sheds light on the general case of the  $d$ -fold free product of graphs, in the same way as the  $d$ -dimensional cube was the leading example for investigations of the distance- $k$  graph of the  $d$ -fold Cartesian product of graphs (Kurihara [9]).

As a warm up and base case, we calculate the distribution of the distance-2 graph with respect to the vacuum state.

For  $d \geq 2$ , let  $A_d^{[k]}$  be the adjacency matrix of distance- $k$  graph of  $d$ -regular tree. We will sometimes omit the subindex  $d$  in the notation and write  $A^{[1]} = A$ . Then we have the following equality, which expresses  $A^2$  in terms of the distance-2 graph and the identity matrix (see Figure 1). :

$$(3.1) \quad A^2 = A_d^{[2]} + dI.$$

Since  $A_d^{[2]} = A^2 - dI$  then the distribution is given by the law of  $x^2 - d$ , where  $x$  is a random variable obeying the Kesten-McKay distribution of parameter  $d$ ,  $\mu_d$ .

For  $k \geq 2$  we have the following recurrence formula.

**Lemma 3.1.** *Let  $d \geq 1$  fixed, then  $A^{[1]} = A$ ,  $A^{[2]} = A^2 - dI$ , and*

$$(3.2) \quad AA^{[k]} = A^{[k+1]} + (d-1)A^{[k-1]} \quad k = 2, \dots, d-1.$$

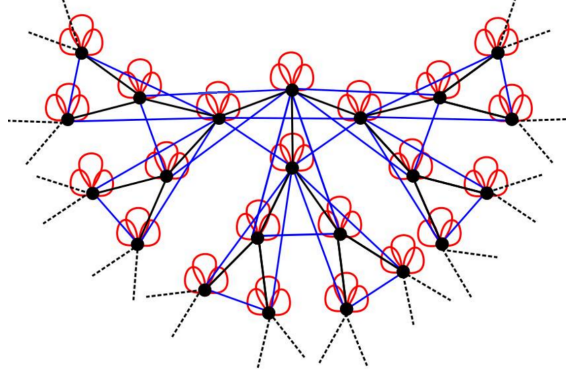


FIGURE 1. Graph of  $A^2$  split in two parts  $A^2 = A_d^{[2]} + dI$ .

*Proof.* Let  $i$  and  $j$  be vertices of the  $d$ -regular tree,  $Y_d$ . We have the following three cases.

**Case 1.** If  $\partial(i, j) = k + 1$  then  $(A^{[k]}A)_{ij} = 1$ , that is because, in this case, there is only one way to get from vertex  $j$  to vertex  $i$ . Indeed, since this  $Y_d$  is a tree there is only one walk from  $i$  to  $j$  of size  $k + 1$  in  $Y_d$ . Thus, there is exactly one neighbor  $l$  of  $j$  at distance  $k$  from  $i$  and thus the only way to go across the distance- $k$  graph and after across  $Y_d$  to reach  $j$  is through  $l$ .

**Case 2.** When we have  $\partial(i, j) = k - 1$ , then  $(A^{[k]}A)_{ij} = d - 1$ . In fact, for the vertex  $i$  there are  $d - 1$  ways to arrive to  $j$  from a neighbor of  $j$  at distance  $k$  from  $i$ . Thus, if we are in vertex  $i$ , there are  $d - 1$  ways to travel across the distance- $k$  graph and finally go down one level in the  $d$ -regular tree to vertex  $j$ .

**Case 3.** Suppose  $|\partial(i, j) - k| \neq 1$ , then  $(A^{[k]}A)_{ij} = 0$ . To see this, we just note that, in the  $d$ -regular tree we can go up one-level or go down one-level, after going across the distances- $k$  graph, this means that the distance between  $i$  and  $j$  would be  $k - 1$  or  $k + 1$ , which is a contradiction. Therefore if  $|\partial(i, j) - k| \neq 1$ , there is no way to go from the vertex  $i$  to the vertex  $j$ , going across the distance- $k$  graph and after, across the  $d$ -regular tree in one step.

Thanks to the above, we obtain the following recurrence formula

$$(3.3) \quad A^{[k]}A = A^{[k+1]} + (d - 1)A^{[k-1]}.$$

From the equations (3.1) and (3.3) we can see that  $A^{[k]}$  is a polynomial in  $A$  for  $k \geq 1$ , and thus commutes with  $A$ . Then we can rewrite equation (3.3) in the more convenient way as follows

$$AA^{[k]} = A^{[k+1]} + (d - 1)A^{[k-1]}.$$

□

Now we can calculate the distribution of the distance- $k$  graph of the  $d$ -regular tree, for  $d$  fixed, which is exactly Theorem 1.1.

*Proof of Theorem 1.1.* From equation (3.2) we can see that  $A_d^{[k]}$  fulfills the same recurrence formula than  $T_k$  in (2.5). Since  $A$  is distributed as the Kesten-McKay distribution  $\mu_d$ , we arrive to the conclusion. □

To end this section we observe that from the considerations above, by letting  $d$  approach infinity, we may find the asymptotic behavior of the distribution of the distance- $k$  graph of the  $d$ -regular tree. The same behavior is expected when changing the  $d$ -regular tree with the  $d$ -fold free product of any finite graph. We will prove this in Section 5 of the paper.

**Theorem 3.2.** *For  $d \geq 2$ , let  $A_d^{[k]}$  be the adjacency matrix of the distance- $k$  graph of the  $d$ -regular tree. Then the distribution with respect to the vacuum state of  $d^{-k/2}A_d^{[k]}$  converges in moments as  $d \rightarrow \infty$  to the probability distribution of*

$$(3.4) \quad P_k(s),$$

where  $P_k(s)$  is the Chebychev polynomial of degree  $k$  and  $s$  is a random variable obeying the semicircle law.

*Proof.* If we divide the equation (3.2) by  $d^{(k+1)/2}$  we obtain

$$\frac{A_d}{d^{1/2}} \frac{A_d^{[k]}}{d^{k/2}} = \frac{A_d^{[k+1]}}{d^{(k+1)/2}} + \frac{A_d^{[k-1]}}{d^{(k-1)/2}} - \frac{1}{d} \frac{A_d^{[k-1]}}{d^{(k-1)/2}}$$

We write  $X = \frac{A_d}{d^{1/2}}$ , then we have

$$P^{(1)}(X) = X, \quad P^{(2)}(X) = X^2 - I,$$

and the recurrence

$$XP^{(k)}(X) = P^{(k+1)}(X) + P^{(k-1)}(X) - \frac{1}{d}P^{(k-1)}(X),$$

which when  $d \rightarrow \infty$  becomes the recurrence formula

$$XP^{(k)}(X) = P^{(k+1)}(X) + P^{(k-1)}(X).$$

Thus  $P^{(k)}(x)$  and  $P_k(x)$  satisfy the same recurrence formula asymptotically and thanks to the free central limit theorem for graphs (Theorem 2.5) we have the convergence,  $X \xrightarrow{m} s$ . Consequently, combining these two observations and using Lemma 2.4 we obtain the proof.  $\square$

#### 4. DISTANCE-2 GRAPH OF FREE PRODUCTS

In this section we derive the asymptotic spectral distribution of the distance-2 graph of the  $n$ -free power of a graph when  $n$  goes to infinity.

In order to analyze the distance-2 graphs we give a simple, but useful, decomposition of the square of the adjacency matrix.

**Lemma 4.1.** *Let  $G$  be a simple graph with adjacency matrix  $A$ , we have the following decomposition of  $A^2$ :*

$$(4.1) \quad A^2 = \tilde{A}^{[2]} + D + \Delta,$$

where  $D$  is diagonal with  $(D)_{ii} = \deg(i)$ ,  $(\Delta)_{ij} = |\text{triangles in } G \text{ with one side } (i, j)|$  and  $(\tilde{A}^{[2]})_{ij} = |\text{paths of size 2 from } i \text{ to } j|$ , whenever  $(A^{[2]})_{ij} = 1$  and  $(\tilde{A}^{[2]})_{ij} = 0$  if  $(A^{[2]})_{ij} = 0$ .

*Proof.* Indeed  $(A^2)_{ij}$  is zero if the distance between  $i$  and  $j$  is greater than 2. So  $(A^2)_{ij} > 0$  implies that  $\partial(i, j) = 0, 1$  or  $2$ . If  $\partial(i, j) = 0$  then  $i = j$  and since  $(A^2)_{ii} = \deg(i)$  we get  $D$ , a diagonal matrix with  $(D)_{ii} = \deg(i)$ . Next, if  $\partial(i, j) = 1$  then each path of size 2 which forms a triangle with side  $(i, j)$  will contribute to  $(A^2)_{ij} = (\Delta)_{ij}$  where  $(\Delta)_{ij} = |\text{triangles in } G \text{ with one side } (i, j)|$ .



Finally if  $\partial(i, j) = 2$  then  $(A^2)_{ij}$  equals the number of paths of size 2 from  $i$  to  $j$ , which is non-zero exactly when  $(\tilde{A}^{[2]})_{ij} > 0$ .  $\square$

**Remark 4.2.** Notice in Lemma 4.1, that when  $G$  is a tree then  $\Delta = 0$ ,  $\tilde{A}^{[2]} = A^{[2]}$ , therefore  $A^{[2]} = A^2 - D$ .

Let  $G = (V, E, e)$  be a rooted graph,  $A_n = A_{G^{*N}}$  and define  $D_n$  and  $\Delta_n$  by the decomposition (4.1) applied to  $G^{*N} = G * \dots * G$ , i.e.

$$(4.2) \quad A_n^2 = \tilde{A}_n^{[2]} + D_n + \Delta_n.$$

We will describe the asymptotic behavior of each of these matrices. First, we consider the diagonal matrix  $D_n$ .

**Lemma 4.3.**  $D_n/n \rightarrow Ideg(e)$  entrywise and in distribution w.r.t. the vacuum state.

*Proof.* For any  $i \in G_n$   $(D_n)_{ii} = deg_{G_n}(i) = c_i + (n-1)deg(e)$  for some  $0 < c_i < maxdeg(G)$ . Thus,

$$\frac{(D_n)_{ii}}{n} = \frac{c_i}{n} + \frac{(n-1)deg(e)}{n} \rightarrow deg(e).$$

$\square$

In order to consider the other matrices in the decomposition we will use the order  $\succeq$  from Definition 2.6.

**Lemma 4.4.** The mixed moments of  $A_n^2/n$  and  $\Delta_n/n$  asymptotically vanish.

*Proof.* Note that the free product does not generate new triangles other than the ones in copies of the original graph. Thus, for  $c = \max \deg(G)$  the relation  $cA_n \succeq \Delta_n$  holds. Hence, for  $m_1, m_2, \dots, m_s, l_1, l_2, \dots, l_s \in \mathbb{N}$  and  $l_1 > 0$ , from Remark 2.7, we have that

$$\begin{aligned} & \varphi_1 \left[ \left( \frac{A_n^2}{n} \right)^{m_1} \left( \frac{\Delta_n}{n} \right)^{l_1} \dots \left( \frac{A_n^2}{n} \right)^{m_s} \left( \frac{\Delta_n}{n} \right)^{l_s} \right] \\ & \leq c^{\sum_i l_i} \varphi_1 \left[ \left( \frac{A_n^2}{n} \right)^{m_1} \left( \frac{A}{n} \right)^{l_1} \dots \left( \frac{A_n^2}{n} \right)^{m_s} \left( \frac{A}{n} \right)^{l_s} \right]. \end{aligned}$$

From Theorem 2.5 we have that  $A^2/n$  and  $A/n^{1/2}$  converge, then the right hand side of the preceding inequality converges to zero as  $n$  goes to infinity.  $\square$

Since  $\tilde{A}_n^{[2]}$  and  $D_n$  are subgraphs of  $A_n^2$  we have the following.

**Corollary 4.5.** The mixed moments of the pairs  $(\tilde{A}_n^{[2]}/n, \Delta/n,)$  and  $(D_n/n, \Delta/n,)$  asymptotically vanish.

Finally, we consider the matrix  $\tilde{A}^{[2]}$ .

**Lemma 4.6.**  $\tilde{A}_n^{[2]}$  converges to  $A_n^{[2]}$  as  $n$  goes to infinity.

*Proof.* Observe that we can write  $A_n^{[2]}$  as

$$\tilde{A}_n^{[2]} = A_n^{[2]} + \square_n,$$

where for  $(i, j)$  at distance 2 in  $G^{*n}$ , the entry  $(\square_n)_{ij}$  exceeds in one the number of vertices  $k$  such that  $i \sim k$  and  $k \sim j$ .

We will extend  $G$  in the following way. For each  $(i, j)$  such that  $\square_{ij}$  is positive we put the edge  $ij$  and call this new graph  $G(ext)$ . Now notice that, by construction,  $\Delta_{G(ext)^{*n}} \succeq \square$  and  $A_{G(ext)^{*n}} \succeq A_{G^*n}$ . Finally, by the previous lemma the mixed moments of  $\Delta_{G(ext)^{*n}}$  and  $A_{G(ext)^{*n}}^2$  asymptotically vanish. But  $A_{G(ext)^{*n}}^2 \succeq A_n^{[2]}$ , so the mixed moments of  $A_n^{[2]}$  and  $\square_n$  also vanish in the limit. This of course means that  $\tilde{A}_n^{[2]}$  and  $A_n^{[2]}$  are asymptotically equal in distribution.  $\square$

We have shown that asymptotically  $D_n/n$  approximates  $I$ ,  $\tilde{A}_n^{[2]}$  approximates  $A_n^{[2]}$  and that the joint moments between  $\tilde{A}_n^{[2]}$  or  $D_n$  and  $\Delta_n$  vanish. Thus, we arrive to the following theorem.

**Theorem 4.7.** *The asymptotic distributions of distance-2 graph of the  $n$ -fold free product graph, as  $n$  tends to infinity, is given by the distribution of  $s^2 - 1$ , where  $s$  is a semicircle.*

*Proof.* From the equation (4.2), and thanks to Lemmas 2.4, 4.3, 4.6, Corolary 4.5 and Theorem 2.5 we have

$$A_n^{[2]} \xrightarrow{D} \tilde{A}_n^{[2]} \xrightarrow{D} A_n^2 - D_n - \Delta_n \xrightarrow{D} A_n^2 - I \xrightarrow{D} s^2 - 1.$$

$\square$

## 5. DISTANCE- $k$ GRAPHS OF FREE PRODUCTS

This section contains the proof of Theorem 1.4 which describes the asymptotic behavior of the distance- $k$  graph of the  $d$ -fold free power of graphs. We will show that the adjacency matrix satisfies in the limit the recurrence formula (2.3). We start by showing a decomposition similar to the one seen above for  $d$ -regular trees which plays the role of Lemma 4.1 in the last section.

**Theorem 5.1.** *Let  $G$  be a simple finite graph, let  $N, k \in \mathbb{N}$  with  $N \geq 2$  and  $k \geq 3$  and let  $A = A_N$  denote the adjacency matrix of  $G^{*N}$ . Then, we have the following recurrence formula*

$$(5.1) \quad A^{[k]}A = \tilde{A}^{[k+1]} + (N-1)\deg(e)A^{[k-1]} + D_N^{[k-1]} + \Delta_N^{[k]},$$

where  $(\tilde{A}^{[k+1]})_{ij} = |\{l \sim j : \partial(i, l) = k\}|$  whenever  $\partial(i, j) = k + 1$ ,  $(D_N^{[k-1]})_{ij} = |\{l \sim j : \partial(i, l) = k, \text{ and } j \text{ and } l \text{ are in the same copy of } G\}|$  if  $\partial(i, j) = k - 1$  and  $(\Delta_N^{[k]})_{ij} = |\{l \sim j : \partial(i, l) = k\}|$  when  $\partial(i, j) = k$ .

*Proof.* It's easy to see that  $(A^{[k]}A)_{ij}$  is zero if  $|\partial(i, j) - k| \geq 2$ . So  $(A^{[k]}A)_{ij} > 0$  implies that  $\partial(i, j) = k - 1$ ,  $k$  or  $k + 1$ .

Notice that for each neighbor  $l$  of  $j$  at distance  $k$  from  $i$ , there is one edge from  $i$  to  $l$  in  $A^{[k]}$  and one from  $l$  to  $j$  in  $A$ . Thus each of these neighbors adds 1 to  $(A^{[k]}A)_{ij}$  and there is no further contribution.

First, if  $\partial(i, j) = k - 1$  there are two types of neighbors  $l$  at distance  $k$  in  $G^{*N}$ . The first ones come from the  $(N - 1)$  copies of  $G$  in  $G^{*N}$  which have  $j$  as a root and contribute to the matrix  $A^{[k-1]}$  by  $(N - 1)\deg(e)$  and the second ones in which  $j$  is in the same copy that  $l$ , which contribute to  $D_N^{[k-1]}$ .

Secondly, if  $\partial(i, j) = k$  and  $(A^{[k]}A)_{ij} > 0$  is the number of neighbors of  $j$  which are at distance  $k$  from  $i$ , then we get  $\Delta_N^{[k]}$ .

Finally, if we have  $\partial(i, j) = k + 1$ , so there exists at least one minimal path from  $i$  to  $j$ , which contains itself a neighbor of  $j$  which is at distance  $k$  from  $i$ , therefore this path contributes to  $\tilde{A}^{[k+1]}$ .  $\square$

**Proof of Theorem 1.4.** We now proceed to prove Theorem 1.4 in various steps. While the steps are very similar as the one for the case  $k = 2$  there are some non trivial modifications to be done for the general case.

We will use induction over  $k$ . First, observe that for  $k = 2$ , we obtained the conclusion in the last section. Now, suppose that the fact holds for all  $l \leq k$ . In order to complete the proof we need the following lemmas and corollaries.

**Lemma 5.2.** *The mixed moments of  $A^{[k]}A/N^{\frac{k+1}{2}}$  and  $\Delta_N^{[k]}/N^{\frac{k+1}{2}}$  asymptotically vanish.*

*Proof.* By definition, since the free product does not generate new cycles,

$$\Delta_N^{[k]} \preceq \max \deg(G) A^{[k]}.$$

Hence, for  $m_1, m_2, \dots, m_s, n_1, n_2, \dots, n_s \in \mathbb{N}$  and  $n_1 > 0$

$$\begin{aligned} & \varphi_1 \left( \left( \frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)^{m_1} \left( \frac{\Delta_N^{[k]}}{N^{\frac{k+1}{2}}} \right)^{n_1} \cdots \left( \frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)^{m_l} \left( \frac{\Delta_N^{[k]}}{N^{\frac{k+1}{2}}} \right)^{n_l} \right) \\ & \leq (\max \deg)^{\sum_i n_i} \varphi_1 \left( \left( \frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)^{m_1} \left( \frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)^{n_1} \cdots \left( \frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)^{m_l} \left( \frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)^{n_l} \right). \end{aligned}$$

By induction hypothesis  $\left( \frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)$  and  $\left( \frac{A^{[k]}A}{N^{\frac{k}{2}}} \right)$  converge and, therefore, the right hand side of the last inequality goes to zero.  $\square$

Since  $\tilde{A}^{[k+1]}$  and  $D_N^{[k-1]}$  are subgraphs of  $A^{[k]}A$ , the following is a direct consequence of the previous lemma.

**Corollary 5.3.** *The mixed moments of  $\left( \tilde{A}^{[k+1]}/N^{\frac{k+1}{2}}, \Delta_N^{[k]}/N^{\frac{k+1}{2}} \right)$  and  $\left( \Delta_N^{[k]}/N^{\frac{k+1}{2}}, D_N^{[k-1]}/N^{\frac{k+1}{2}} \right)$  asymptotically vanish.*

**Corollary 5.4.** *The matrices  $\Delta_N^{[k]}/N^{\frac{k+1}{2}}$  and  $D_N^{[k-1]}/N^{\frac{k+1}{2}}$  converge to zero as  $N$  tends to infinity.*

*Proof.* In the proof of Lemma 5.2, by taking  $m_i = 0$  for all  $0 \leq i \leq s$  we obtain the conclusion for  $\Delta_N^{[k]}/N^{\frac{k+1}{2}}$ . Using  $A^{[k-1]}$  instead  $A^{[k]}$  the same proof works for  $D_N^{[k-1]}/N^{\frac{k+1}{2}}$ .  $\square$

In the proof of the next lemma, we shall use the following extension of a graph  $G$ . For  $k \geq 2$  we define  $G_{ext(k)}$  as the graph which contains the graph  $G$ , and if  $G$  has a cycle of even length smaller than  $2k$ , we add all the possible edges between the vertices of this cycle. Note that  $G_{ext(2)} = G_{ext}$ . It is important to notice the fact that

$$(G_{ext(k)})^{*N} = (G^{*N})_{ext(k)}.$$

**Lemma 5.5.** *Let  $k \geq 2$ , then  $\lim_{N \rightarrow \infty} \frac{\tilde{A}^{[k+1]}}{N^{\frac{k+1}{2}}} - \frac{A^{[k+1]}}{N^{\frac{k+1}{2}}} = 0$ .*

*Proof.* Let  $i, j \in \bigcup_{s \in [N]}^* V$  be such that  $\left(\tilde{A}^{[k+1]}\right)_{ij} > 0$ . Let

$$C_{ij}^{k+1} = \{\text{cycles of even length in a path of length } k+1 \text{ from } i \text{ to } j\},$$

notice that

$$|C_{ij}^{k+1}| \leq (\max \deg(G))^{k+1}.$$

Here, is important to observe that the right side bound does not depend on  $i, j$  neither  $N$ , because the free product of graph does not produce new cycles. Then we can write

$$(5.2) \quad \tilde{A}^{[k+1]} - A^{[k+1]} \preceq (\max \deg(G))^{k+1} \left( A_{G_{ext(k+1)}}^{[k]} + A_{G_{ext(k+1)}}^{[k-1]} + \cdots + A_{G_{ext(k+1)}} \right).$$

Then, we obtain from (5.2)

$$\begin{aligned} & \left( \frac{\tilde{A}^{[k+1]} - A^{[k+1]}}{N^{(k+1)/2}} \right) \\ & \preceq (\max \deg(G))^{k+1} \left( \frac{A_{G_{ext(k+1)}}^{[k]}}{N^{(k+1)/2}} + \frac{A_{G_{ext(k+1)}}^{[k-1]}}{N^{(k+1)/2}} + \cdots + \frac{A_{G_{ext(k+1)}}}{N^{(k+1)/2}} \right) \\ & = (\max \deg(G))^{k+1} \left( \frac{A_{G_{ext(k+1)}}^{[k]}}{N^{\frac{k}{2}}} \frac{1}{N^{1/2}} + \frac{A_{G_{ext(k+1)}}^{[k-1]}}{N^{\frac{k-1}{2}}} \frac{1}{N} + \cdots + \frac{A_{G_{ext(k+1)}}}{N^{\frac{1}{2}}} \frac{1}{N^{\frac{k}{2}}} \right). \end{aligned}$$

By induction hypothesis we have that  $\left( A_{G_{ext(k)}}^{[i]} / N^{i/2} \right)$  converges for all  $i \leq k$ . Therefore we have

$$\left( \frac{\tilde{A}^{[k+1]} - A^{[k+1]}}{N^{(k+1)/2}} \right) \xrightarrow{N \rightarrow \infty} 0,$$

which completes the proof.  $\square$

Now we can finish the proof of Theorem 1.4. From (5.1) we have that

$$(5.3) \quad \frac{A_N^{[k+1]}}{(\deg(e)N)^{\frac{k+1}{2}}} = \frac{A_N^{[k]} A_N}{(\deg(e)N)^{\frac{k+1}{2}}} - \frac{A_N^{[k-1]}}{(\deg(e)N)^{\frac{k-1}{2}}} - C(N, k+1),$$

where

$$C(N, k+1) = \frac{\deg(e)A_N^{[k-1]} + \Delta_N^{[k]} + D_N^{[k-1]} - \left( \tilde{A}^{[k+1]} - A^{[k-1]} \right)}{(\deg(e)N)^{\frac{k+1}{2}}}.$$

Due to the induction hypothesis we have,  $\deg(e)A_N^{[k-1]} / (\deg(e)N)^{\frac{k+1}{2}}$  converging to zero, furthermore by Corollary 5.4 and Lemma 5.5

$$\frac{\Delta_N^{[k]} + D_N^{[k-1]} - \left( \tilde{A}^{[k+1]} - A^{[k-1]} \right)}{(\deg(e)N)^{\frac{k+1}{2}}},$$

converges to zero. Hence

$$(5.4) \quad C(N, k+1) \longrightarrow 0.$$

Finally, using the induction hypothesis we can see that

$$(5.5) \quad \frac{A_N^{[k]} A_N}{(\deg(e)N)^{\frac{k+1}{2}}} - \frac{A_N^{[k-1]}}{(\deg(e)N)^{\frac{k-1}{2}}} \longrightarrow P_k(s)s - P_{k-1}(s) = P_{k+1}(s),$$

where the last equality is given by (2.3). Thus, mixing (5.3) with (5.4) and (5.5) we obtain that

$$\frac{A_N^{[k+1]}}{(\deg(e)N)^{\frac{k+1}{2}}} \longrightarrow P_{k+1}(s).$$

## 6. $d$ -REGULAR RANDOM GRAPHS

Apart from the Erdos-Renyi models [5, 6], possibly, the random  $d$ -regular graphs are possibly the most studied and well understood random graphs.

In the original paper by McKay [13], he proved that the asymptotical spectral distributions of  $d$ -regular random graph are exactly the ones appearing in (2.4). Heuristically, the reason is that, locally, large random  $d$ -regular graphs look like the  $d$ -regular tree and thus asymptotically their spectrum should coincide. This turns out to remain true for the distance- $k$  graph and thus we shall expect to get a similar result. In this section we formalize this intuition.

Let  $X$  be a  $d$ -regular graph with vertex set  $\{1, 2, \dots, n(X)\}$ . For each  $i \geq 3$  let  $c_i(X)$  be the number of cycles of length  $i$ . Let  $A^{[k]}(X)$  be the adjacency matrix of the distance- $k$  graph of  $X$ . The following is a generalization of the main theorem in McKay [13].

**Theorem 6.1.** *For  $d \geq 2$  fixed, let  $X_1, X_2, \dots$  be a sequence of  $d$ -regular graphs such that  $n(X_i) \rightarrow \infty$  and  $c_j(X_i)/n(X_i) \rightarrow 0$  as  $i \rightarrow \infty$  for each  $j \geq 3$ . Then the distribution with respect to the normalized trace of  $A^{[k]}(X_i)$  converges in moments, as  $i \rightarrow \infty$ , to the distribution of  $A_d^{[k]}$  with respect to the vacuum state.*

*Proof.* We follow the original proof of McKay[13] with simple modifications. Let  $n_r(X_i)$  denote the number of vertices  $v$  of  $X_i$  such that the subgraph of  $X_i$  induced by the vertices at distance at most  $r = mk$ , where  $m \in \mathbb{N}$ , from  $v$  is acyclic. By hypothesis we have that  $n_r(X_i)/n(X_i) \rightarrow 1$  as  $i \rightarrow \infty$ . The number of closed walks of length  $m$  in the distance- $k$  graph of the  $d$ -regular graph starting at each such vertex is  $\varphi(A_d^{[k]})$ . For each of the remaining vertices the number of closed walks of length  $m$  is certainly less than  $d^r$ . Then, for each  $m$ , there are numbers  $\hat{\varphi}_m(X_i)$  such that  $0 \leq \hat{\varphi}_m(X_i) \leq d^r$ , and

$$\begin{aligned} \varphi_{tr} \left( A^{[k]}(X_i) \right) &= \frac{\varphi(A_d^{[k]})n_r(X_i)}{n(X_i)} + \frac{(n(X_i) - n_r(X_i))\hat{\varphi}_m(X_i)}{n(X_i)} \\ &\longrightarrow \varphi(A_d^{[k]}) \quad \text{as } i \rightarrow \infty. \end{aligned}$$

□

Fix  $d > 0$ . Let  $s_1 < s_2 < \dots$  be the sequence of possible cardinalities of regular graphs with degree  $d$ . For each  $n$  define  $R_n$  to be the set of all labeled regular graphs with degree  $n$  and order  $s_i$ .

In order to consider the  $d$ -regular uniform random graphs we use the following lemma of Wormald [17].

**Lemma 6.2.** *For each  $k > 3$  define  $c_{k,n}$  to be the average number of  $k$ -cycles in the members of  $R_n$ . Then for each  $k$ ,  $c_{k,n} \rightarrow (d-1)^k/2k$  as  $n \rightarrow \infty$ .*

*Proof of Theorem 1.5.* Consider a graph  $G_n$  which consist a disjoint union of the all the labelled graphs of size  $s_n$ . The eigenvalue distribution of  $G_n$  coincides with the expected eigenvalue distribution of  $R_n$ . Now by Lemma 6.2,  $G$  satisfies the assumptions of Theorem 6.1 and thus we arrive to the theorem.  $\square$

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